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Square-roots, Internal Symmetries, and Fermion Field Quantization

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Abstract

It is observed that the complex square-root of the hermitian matrix $\sigma^\mu p_\mu$ associated with a physical four-momentum admits Lorentz-independent unitary transformations that may be related to the internal symmetries of hadrons. An operator-valued square-root of the Hilbert space inner product in relativistic one-electron theory brings in fermion field quantization conditions as a direct concomitant of its linearity and hermicity properties.

Dirac's square-root problem of 1928, that of expressing the Lorentz invariant $(p_\mu p^\mu)^{1/2} \equiv [(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2]^{1/2}$ as a vector-space operator linear in the components of the four-momentum $p = (p_0, p_1, p_2, p_3)$, was central to the formulation of the relativistic electron wave equation (see, for example, Schweber, 1961, Chap. 4). In analogy to Dirac's $(p_\mu p^\mu)^{1/2}$ problem, there are two fundamental square-root problems in relativistic particle physics which may enter importantly in future theories but which apparently have not been noted and discussed heretofore in the literature. The purpose of the present communication is to bring these two square-root problems and their simple solutions to the attention of theoretical physicists, with the aim of stimulating further thought and constructive research about them.

Consider the 2×2 hermitian matrix

$$\sigma^\mu p_\mu \equiv \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \tag{1}$$

formed by contraction of the 2×2 identity and Pauli matrices with a four-momentum p. Under a homogeneous orthochronous Lorentz transformation

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$p_\mu \xrightarrow{\Lambda} p'_\mu = \Lambda_\mu{}^\nu p_\nu$, it is well known that the matrix (1) is transformed by an element of the $SL(2, c)$ covering group,

$$\sigma^\mu p_\mu \xrightarrow{\Lambda} T(\Lambda)\sigma^\mu p_\mu T(\Lambda)^\dagger \tag{2}$$

where $T(\Lambda)$ is a 2×2 complex unimodular matrix [$\det T(\Lambda) = 1$] determined to within a ± 1 multiplying factor by $\Lambda = (\Lambda_\mu{}^\nu)$, and $T(\Lambda)^\dagger$ is its hermitian adjoint. Thus, the determinant of (1)

$$\det(\sigma^\mu p_\mu) = p^\mu p_\mu = m^2 (\geqslant 0) \tag{3}$$

is an invariant non-negative constant (the mass-squared of the particle) under orthochronous Lorentz transformations. For a physical particle of positive energy, the trace of (1)

$$\operatorname{tr}(\sigma^\mu p_\mu) = 2p_0 (> 0) \tag{4}$$

is a positive quantity that changes in magnitude under the Lorentz transformation $SL(2, c)$ representation in (2). Because both the determinant and the trace of the hermitian matrix (1) are non-negative, the eigenvalues of (1) $[p_0 \pm (p_0{}^2 - m^2)^{1/2}]$ are non-negative quantities, and an elementary theorem of matrix algebra (see, for example, Perlis, 1952) guarantees that the matrix (1) can be expressed generally in the factorized form

$$\sigma^\mu p_\mu = \Gamma(p)\Gamma(p)^\dagger \tag{5}$$

where $\Gamma(p)$ is a 2×2 complex matrix that depends algebraically on the components of the four-momentum p, and $\Gamma(p)^\dagger$ is its hermitian adjoint. In order for the complex square-root factorization of $\sigma^\mu p_\mu$ in (5) to be consistent with (2), the matrix $\Gamma(p)$ must transform

$$\Gamma(p) \xrightarrow{\Lambda} T(\Lambda)\Gamma(p) \tag{6}$$

under an orthochronous Lorentz transformation. However the matrix $\Gamma(p)$ is not determined uniquely by its progenitor $\sigma^\mu p_\mu$, and the factorization (5) is invariant with respect to certain transformations of $\Gamma(p)$ that are wholly independent of the Lorentz frame, namely transformations of the form

$$\Gamma(p) \longrightarrow \Gamma(p)\mathscr{U} \tag{7}$$

with $\mathscr{U}$ an arbitrary 2×2 unitary matrix, $\mathscr{U}\mathscr{U}^\dagger = 1$. If the transformations (7) were to be associated with the $U(2)(= SU(2) \oplus U(1))$ isospin-baryon number internal symmetry group, the factorization (5) would yield a purely kinematic explanation for the primary internal symmetries of hadrons. Moreover, the factorization (5) admits consistent interpretation with $\Gamma(p)$ generalized and defined as a $2 \times n$ complex rectangular matrix array and the internal symmetry unitary matrix in (7) defined as $n \times n$. Thus, a complex square-rooting of $\sigma^\mu p_\mu$ can induce $SU(3)$ and higher internal symmetries with appropriate definition of the dimensionality of $\Gamma(p)$ in (5). For a system of k

particles which undergo a scattering process, conservation of total four-momentum

$$\sum_{j=1}^{k} p_{\text{in}}^{(j)} = \sum_{j=1}^{k} p_{\text{out}}^{(j)} \tag{8}$$

implies that the associated $\Gamma(p)$'s are related by the unitary transformation formula (for $\Gamma(p_{\text{out}}^{(j)})$ expressed linearly in terms of the $\Gamma(p_{\text{in}}^{(j')})$'s),

$$\sum_{j=1}^{k} \Gamma(p_{\text{in}}^{(j)})\Gamma(p_{\text{in}}^{(j)})^{\dagger} = \sum_{j=1}^{k} \Gamma(p_{\text{out}}^{(j)})\Gamma(p_{\text{out}}^{(j)})^{\dagger} \tag{9}$$

while (6) implies that the elements of the matrices $\Gamma(p_{\text{in}}^{(j)})^{-1}\Gamma(p_{\text{in}}^{(j')})$, $(j, j' = 1, \ldots, k)$, are Lorentz-invariant parameters for the scattering.

The second square-root problem concerns the Hilbert space inner product in relativistic one-electron theory,

$$(\xi, \eta) \equiv \int \sum_{\alpha=1}^{4} \xi_{\alpha}^{*}(\mathbf{x})\eta_{\alpha}(\mathbf{x})\, d^3x \tag{10}$$

where ξ and η are four-component complex-valued Dirac spinor wave functions (with a possible time-dependence suppressed). Although the inner product (10) is linear in η and antilinear in ξ, the double-valued complex number $(\xi, \eta)^{1/2}$ does not feature any simple linearity property. Nevertheless, it is possible to define an operator $\Omega_{\xi\eta}$ which is linear on the direct-sum space of eight-component complex-valued functions $\xi^* \oplus \eta$, satisfies the hermicity condition $\Omega_{\xi\eta}^{*} = \Omega_{\eta\xi}$, and squares to the quantity (10),

$$(\xi, \eta) = (\Omega_{\xi\eta})^2 \tag{11}$$

By introducing the Dirac spinor field operator $\boldsymbol{\psi} = (\boldsymbol{\psi}_{\alpha}(\mathbf{x}))$ and the prescribed form

$$\Omega_{\xi\eta} = (\xi, \boldsymbol{\psi}) + (\boldsymbol{\psi}, \eta) \tag{12}$$

it follows that (12) satisfies (11) if and only if $\boldsymbol{\psi}$ satisfies the equal-time fermion field quantization conditions (see, for example, Schweber, 1961, p. 226)

$$\begin{aligned} &\boldsymbol{\psi}_{\alpha}(\mathbf{x})\boldsymbol{\psi}_{\beta}(\mathbf{x}') + \boldsymbol{\psi}_{\beta}(\mathbf{x}')\boldsymbol{\psi}_{\alpha}(\mathbf{x}) = 0 \\ &\boldsymbol{\psi}_{\alpha}^{*}(\mathbf{x})\boldsymbol{\psi}_{\beta}^{*}(\mathbf{x}') + \boldsymbol{\psi}_{\beta}^{*}(\mathbf{x}')\boldsymbol{\psi}_{\alpha}^{*}(\mathbf{x}) = 0 \\ &\boldsymbol{\psi}_{\alpha}(\mathbf{x})\boldsymbol{\psi}_{\beta}^{*}(\mathbf{x}') + \boldsymbol{\psi}_{\beta}^{*}(\mathbf{x}')\boldsymbol{\psi}_{\alpha}(\mathbf{x}) = \delta^{(3)}(\mathbf{x} - \mathbf{x}')\delta_{\alpha\beta} \end{aligned} \tag{13}$$

Thus, fermion field quantization is concomitant with the existence of an operator-valued square-root of the inner product (10) which features the linearity and hermicity properties manifest in (12). This suggests a central role for $\Omega_{\xi\eta}$ in a future theory for elementary particles, with the fermion quantum field operator *derived* from the quantity (12) by functional differentiation (see, for example, Rosen, 1969),

$$\delta\Omega_{\xi\eta}/\delta\xi_{\alpha}^{*}(\mathbf{x}) = \Psi_{\alpha}(\mathbf{x}), \qquad \delta\Omega_{\xi\eta}/\delta\eta_{\alpha}(\mathbf{x}) = \Psi_{\alpha}^{*}(\mathbf{x}) \tag{14}$$

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